ON LIMIT APERIODIC G-SETS

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Abstract. We prove that the property to be limit aperiodic is preserved by the standard construction with groups like extension, HNN extension and free product. We also construct a non-limit aperiodic G-space, give the definition of limit aperiodic subgroups and explore their properties.

1. Introduction

If a discrete group G acts by isometries freely and cocompactly on a metric space X one can study periodic and aperiodic tilings of X. A tiling of X can be defined first as a tiling with one tile, the Voronoi cell (see [1]). Using a finite set of colors one can consider tilings of X by color. Then using the "notching" one can switch from a tiling by color to a geometric tiling. The standard example here is $G = \mathbb{Z}^2$ and $X = \mathbb{R}^2$. Note that the group G in the above tilings is in bijection with the tiles. Thus, construction of a geometric tiling on X can be reduced to a coloring of the group G. In this paper we study the colorings of discrete groups G that lead to limit aperiodic tilings.

Let $b \in G$, a coloring ϕ of a group G is b-periodic if it is invariant under translation by b, i.e., for every element $g \in G$ the elements g and bg have the same color. A coloring ϕ is aperiodic if it is not b-periodic for any $b \in G \setminus \{e\}$. This can be rephrased as"The stabilizer of ϕ in the space of all colorings of G is trivial". For infinite groups there is a strong notion of periodicity: A coloring ϕ is strongly G-periodic if $|Orb_G(\phi)| < \infty$. The corresponding negation called 'weakly aperiodic' means that the orbit $Orb_G(\phi) = G/Stab_G(\phi)$ of ϕ is finite. A coloring ϕ is called (weakly) limit aperiodic if all colorings in the closure of the orbit $\overline{Orb_G(\phi)}$ taken in an appropriate space of all colorings are (weakly) aperiodic.

In this paper we consider the question raised in [1]: *Which groups admit limit aperiodic colorings by finitely many colors?* This is not obvious question even for $G = \mathbb{Z}$. In [1] it was answered positively for torsion free hyperbolic groups, Coxeter groups, and groups comensurable to them.

This question can be stated in terms of Topological Dynamical Systems theory: *Let G be a group and F be a finite set. Does the natural action of G on the Cantor set F^G admit a G-invariant compact subset $X \subset F^G$ such that the action of G on X is free?* The dynamical system reformulation of a corresponding question about limit weak aperiodic colorings asks about a G-invariant compact subset $X \subset F^G$ such that the orbits $Orb_G(x)$ are infinite for all $x \in X$. This was answered affirmatively by V. Uspenskii [1]. Moreover, E. Glasner proved that there is a minimal set $X \subset F^G$ and $x \in X$ with the trivial stabilizer $Stab_G(x) = e$. In this paper we give a group theoretic approach. We call a group 'limit aperiodic' (LA for

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short) if it admits a limit aperiodic coloring by finitely many colors. We show that the simple group constructions like the product, the extension, the HNN extension, the amalgamated product and the free product preserve the LA property. To prove these facts we introduce the notion of LA G-space and prove the action theorem. In the end of the paper we show that the main question has a negative answer for a specific G-set (the natural numbers) where G is a f.g. subgroup of $Aut(\mathbb{Z})$. In the last section we introduce the notion of a limit aperiodic subgroup and explore some of it's properties in order to investigate whether a space is a limit aperiodic G-space. While this paper was finalized Su Gao, Steven Jackson and Brandon Seward announced their paper : A coloring property of countable groups in which they prove that all countable groups have the coloring property. A preprint can be found here: http://www.cas.unt.edu/ sgao/pub/pub.html

2. Limit Aperiodic Groups

Definition 2.1. Let G be a f.g. group. Also, let F be a finite set of elements which we can think of as colors. A map ϕ from G to F is called a *coloring* of G.

Definition 2.2. Let G, F be as above. We denote by F^G the set of all colorings from G to F. If we consider F with the discrete topology, F^G with the product topology becomes a topological space homeomorphic to the Cantor set.

Definition 2.3. Let G, F as above. Then G acts on F^G with the left action $\delta : G \times F^G \to F^G$ defined by the formula $(g * f)(a) = f(g^{-1} \cdot a)$ for every $g, a \in G$ and $f \in F^G$.

Since F^G is metrizable, a function ϕ belongs to the closure of the orbit of f, $\phi \in \overline{Orb_G(f)}$, if and only if $\phi = \lim \phi_k$, $\{\phi_k\} \subset Orb_G(f)$. This is equivalent to the existence of a sequence $\{h_k\} \subset G$ with $\phi_k = h_k * f$. The condition $\phi = \lim(h_k * f)$ implies that for every $g \in G$ there exists a $k(g) \in \mathbb{N}$ with: $\phi(g) = h_k * f(g)$ for all $k \geq k(g)$.

Definition 2.4. Let G,F as above. A map $f : G \to F$ is called *aperiodic* if the equation $b * f = f$ implies $b = e$.

If the equation $b * f = f$ holds for some $b \in G$ we call f *b-periodic* and b is called a *period of f*.

Definition 2.5. (LA1) Let G, F be as above. A map $f : G \to F$ is called *limit aperiodic* if and only if every $\phi \in \overline{Orb_G(f)}$ is aperiodic.

Definition 2.6. (LA2) Let G, F be as above. A map $f : G \to F$ will be called *limit aperiodic* if for every $g \in G \setminus \{e\}$ there exists a finite set $S \subseteq G$, $S = S(g)$, such that for every $h \in G$ there is a $c \in S$ with $f(hc) \neq f(hgc)$.

Proposition 2.7. *These two definitions are equivalent for finitely generated groups.*

Proof. Suppose that f satisfies the $(LA2)$ property but not the $(LA1)$. Then there exists a $\phi \in \overline{Orb_G(f)}$ such that ϕ has period $g \neq e$. Then g^{-1} is also a period. Let $\{h_k\}_{k \in \mathbb{N}} \in G$ such that $\phi = \lim h_k * f$. Choose the set S for that g. Since S is finite we also have that $g \cdot S$ is finite. From the fact that $\phi = \lim h_k * f$, there exists an $n \in \mathbb{N}$ such that for all $k \geq n$ and for all $x \in S \bigcup g \cdot S$ we have $\phi(x) = h_k * f(x)$. We apply LA2 for f with g and h_n^{-1} to obtain $c \in S$ such that $f(h_n^{-1}c) \neq f(h_n^{-1}gc)$. This contradicts with the fact that g^{-1} is a period for ϕ:

$$\phi(c) = (h_n * f)(c) = f(h_n^{-1}c) \neq f(h_n^{-1}gc) = (h_n * f)(gc) = \phi(gc) = (g^{-1} * \phi)(c).$$

Let's suppose now that f satisfies the (LA1) but not the (LA2). Then there exists a $g \in G$ such that for every finite subset S of G there exists an $h \in G$ with the property:

$$f(hc) = f(hgc)$$

for all $c \in S$.

Fix that $g \in G$. Take $S_1 = \{c \in G : d(c, e) \leq 1\}$ The distance mentioned is the one induced by the word metric in the Cayley graph of G. Since G is f.g. $|S_1| < \infty$, so, there exists an $h_1 \in G$ with $f(h_1 c) = f(h_1 gc)$ for all $c \in S_1$. Take $S_2 = \{c \in G : d(c, e) \leq 2\}$. Again $|S_2| < \infty$. Then there exists an $h_2 \in G$ such that $f(h_2 c) = f(h_2 gc)$ for all $c \in S_2$. Continue for any $k \in \mathbb{N}$.

Thus we obtain a sequence $\{h_k\}_{k \in \mathbb{N}} \in G$. Taking a subsequence we may assume that there is a limit:

$$\phi = \lim_{k \to \infty} h_k^{-1} * f.$$

The claim is that ϕ is periodic with period g. Consider an arbitrary $x \in G$. Name $k_1 = d(x, e)$, then $x \in S_k$ for all $k \geq k_1$. Also since ϕ is the limit of $h_k^{-1} * f$ there exists a $k_2 \in \mathbb{N}$ such that for all $k \geq k_2$:

$$\phi(x) = (h_k^{-1} * f)(x)$$

Finally since ϕ is the limit of $h_k * f$ there exists a $k_3 \in \mathbb{N}$ such that for all $k \geq k_3$:

$$\phi(gx) = (h_k^{-1} * f)(gx)$$

Thus, for $k \geq \max\{k_1, k_2, k_3\}$ we have:

$$\phi(x) = (h_k^{-1} * f)(x) = f(h_k x) = f(h_k gx) = (h_k^{-1} * f)(gx) = \phi(gx) = (g^{-1} * \phi)(x).$$

Since x was taken arbitrarily, we have that ϕ has g^{-1} as a period. This is a contradiction since ϕ belongs to the $\overline{Orb_G(f)}$ and f has the (LA1) property. □

Definition 2.8. A finitely generated group G will be called *limit aperiodic* if it admits a limit aperiodic coloring $f : G \to F$ with a finite set of colors F.

Remark 2.9. *The definition of limit aperiodic groups can easily be extended to any group and not only finitely generated ones. Both the property (LA1) and (LA2) apply to groups without the f.g. hypothesis. Their equivalence though depends on the fact that the group is finitely generated. For us a group (not necessarily finitely generated) will be limit aperiodic if it satisfies the (LA1) property.*

We recall the notion of uniform aperiodicity from [1]. Before we introduce that notion lets establish some notation:

Notation. Let Γ be the Cayley graph of a group G and d be the associated metric. We denote the *displacement* of g at h with:

$$d_g(h) = d(gh, h)$$

With $B_r(h)$ we denote the ball of radius r with center h. Finally $\|g\|$, the norm of g, is the distance between g and e namely:

$$\|g\| = d(g, e)$$

Definition 2.10. Let G be a finitely generated group. A map $f : G \to F$ where F is a finite set (of colors) will be called *uniformly aperiodic* (UA) if there exists a constant $\lambda > 0$ such that for every element $g \in G \setminus \{e\}$ and every $h \in G$, there exists $b \in B_{\lambda d_g(h)}(h)$ with $f(gb) \neq f(b)$.

Definition 2.11. A finitely generated group is called *uniformly aperiodic* if there exists an F and a f as above, so that $f : G \to F$ is uniformly aperiodic.

Proposition 2.12. *If $f : G \to F$ is uniformly aperiodic then f is limit aperiodic.*

Proof. We show that f satisfies LA2. Let $g \in G \setminus \{e\}$ and $h \in G$ arbitrary chosen. Define $S = B_{\lambda\|g\|}(e)$ to be the ball with center e and radius $\lambda\|g\|$.

Clearly since G is finitely generated, S is finite. Assume that there exists an $h \in G$ such that for every $c \in S$ we have $f(hc) = f(hgc)$.

Denote $a = hgh^{-1}$. We apply the UA condition for f with a and h to obtain b in $B_{\lambda d_a(h)}(h)$ with $f(ab) \neq f(b)$. Since $b \in B_{\lambda d_a(h)}(h)$ we have that:

$$d(b, h) \leq \lambda d_a(h) = \lambda d(hgh^{-1}h, h) = \lambda d(hg, h) = \lambda d(g, e) = \lambda\|g\|$$

where the third equality comes from the fact that the metric is left invariant. Notice that:

$$d(h^{-1}b, e) = d(h^{-1}b, h^{-1}h) = d(b, h)$$

Thus $d(h^{-1}b, e) \leq \lambda\|g\|$. This implies that $c = h^{-1}b$ belongs to S. So

$$f(b) = f(h(h^{-1}b)) = f(hc) = f(hgc) = f(hgh^{-1}b) = f(ab)$$

which is clearly a contradiction. □

Remark 2.13. *Let $N = \{a \in G : a$ is a period for $\phi\}$ then $N \lhd G$.*

Proof. Clearly $e \in N$ so $n \neq \emptyset$. Let $a, b \in N$. Then clearly $b^{-1} \in N$ and for every $x \in G$:

$$ab^{-1} * \phi(x) = a * \phi(bx) = \phi(bx) = b^{-1} * \phi(x) = \phi(x)$$

Thus N is a subgroup of G. Let now $g \in G$ and $a \in N$ then for every $x \in G$:

$$\begin{aligned}(gag^{-1} * \phi)(x) &= (ga * \phi)(gx) \\ &= g * (a * \phi(gx)) \\ &= g * \phi(gx) \\ &= \phi(g^{-1}gx) \\ &= \phi(x)\end{aligned}$$

Thus $gag^{-1} \in N$. □

Corollary 2.14. *Let G be a simple group. Then G is limit aperiodic.*

Proof. Let $f : G \to \{0, 1\}$ such that $f(e) = 0$ and $f(x) = 1$ for $x \neq e$. Consider $\phi \in \overline{Orb_G(f)}$ Name $N = \{a \in G : a$ is a period for ϕ. From the remark above N is a normal subgroup of G. Since G is simple we have two possibilities.

Case 1: $N = \{e\}$. This clearly implies that ϕ is aperiodic and thus f is limit aperiodic.

Case 2: $N = G$ Consider $\{g_k\} \subseteq G$, $\phi = \lim g_k * f$. Let $k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$ we have $\phi(e) = g_K * f(e)$. Especially for that g_{k_0} we get:

$$\phi(e) = g_{k_0} * f(e) = f(g_{k_0}^{-1}e)$$

Since $g_{k_0}^{-1} \in G = N$ we have:

$$g_{k_0}^{-1} * \phi(e) = \phi(e) \Rightarrow g_{k_0}^{-1} * f(g_{k_0}^{-1}e) = \phi(e) \Rightarrow f(g_{k_0}e) = \phi(e) \Rightarrow f(e) = \phi(e)$$

So for all $k \geq k_0$ we have $1 = \phi(e) = g_k * f(e) = f(g_k e)$. This implies that $g_k e = g_k = e$ for all $k \geq k_0$. This means that $\phi = f$. Clearly $N = \{a \in G : a$ is a period $f\}$. But the only period of f is $\{e\}$ which is a contradiction. □

3. G-Sets And Limit Aperiodicity

The notion of limit aperiodicity can be generalized in the case of G-Sets. Namely let X be a space and suppose that G acts on X giving it the structure of a G-set. We will use the notation $gx = g(x)$ for $g \in G$ and $x \in X$. Fix a finite set F, which we can consider again as colors.

Denote by F^X the set of all maps from X to F. Then F^X can become a G-set under the following action:

$$(g * f)(x) = f(g^{-1}x)$$

for all $x \in X$, $g \in G$ and $f \in F^X$.

Also denote by: $Fix_G(X) = \{g \in G : g \cdot x = x, \forall\ x \in X\}$ the kernel of the action.

We naturally get the following definitions:

Definition 3.1. Let X be a G-set and let $f \in F^X$. We call f *limit aperiodic* if and only if for every $\phi \in \overline{Orb_G(f)}$ we have that ϕ is aperiodic meaning that if $a * \phi = \phi$, then $a \in Fix(X)$.

Thus we get the definition of limit aperiodic G-sets:

Definition 3.2. Let X be a G-set. If there exists a finite set F and a map

$$f : X \to F$$

such that f is limit aperiodic we say that X is a *limit aperiodic G-set.*

Remark 3.3. *If we consider a group G acting on itself with left multiplication then G is limit aperiodic as a G-set if and only if G is limit aperiodic as a group, because under that action $Fix(G) = \{e\}$.*

Let X be a G-set and let $G_x = Stab_G(x) = \{g \in G \mid gx = x\}$ denote the stabilizer of $x \in X$.

Theorem 3.4. *Let X be limit aperiodic G-set and suppose that G acts transitively on X. Fix $x \in X$ such that $X = Orb_G(x)$. If $Stab_G(x)$ is a limit aperiodic group for some $x \in X$ then G is a limit aperiodic group.*

Proof. Let $\phi : X \to F_1$ be a limit aperiodic map for X and let $\psi : G_x \to F_2$ be a limit aperiodic map for G_x. We know that there exists a bijection π between the set of left cosets G/G_x and the orbit $Orb_G(x)$. Fix a set of representatives in G namely $\{a_i : i \in I\}$ for the quotients G/G_x. Then $\pi(a_j G_x) = a_j x$.

Define $f = (f_1, f_2) : G \to F_1 \times F_2$ by $f_1(g) = \phi(gx)$ and $f_2(g) = \psi(a_j^{-1}g)$ where $g \in a_j G_x$. We will prove that this f is a limit aperiodic map. Suppose that this is false. Then there exists a map $\overline{f} \in \overline{Orb_G(f)}$ and an element $a \in G$ such that $\overline{f}$ has a as a period, i.e., $(a * \overline{f}) = \overline{f}$ for all $x \in X$. Let $\overline{f} = \lim h_k * f$ where $h_k \in G$

$\underline{Case 1)}$ Suppose $a \notin G_x$. Consider the limit

$$\overline{\phi} = \lim_k h_k * \phi.$$

Note that we can always choose a subsequence of h_k such that the limit exists. For convenience we keep the same indices for the subsequence. Clearly, $\overline{\phi} \in \overline{Orb_G(\phi)}$. Given $g \in G$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have:

$$\overline{\phi}(gx) = (h_k * \phi)(gx)$$

Also there exists a $k_1 \in \mathbb{N}$ such that for all $k \geq k_1$ we get: $\overline{f}(g) = (h_k * f)(g)$. Let $k_2 = \max\{k_0, k_1\}$ then for all $k \geq k_2$ we have $\overline{f}(g) = f(h_k^{-1} g)$. Therefore,

$$\overline{f_1}(g) = f_1(h_k^{-1} g) = \phi(h_k^{-1} gx) = (h_k * \phi)(gx) = \overline{\phi}(gx).$$

Following the previous proof and replacing g with $a^{-1}g$ we find a $k_3 \in \mathbb{N}$ such that for all $k \geq k_3$ we have:

$$\overline{f_1}(a^{-1}g) = \overline{\phi}(a^{-1}gx)$$

Since $\overline{f}_1$ has period a we have $(a * \overline{f})(g) = \overline{f}(g)$. Hence,

$$(a * \overline{\phi})(gx) = \overline{\phi}(a^{-1}gx) = \overline{f_1}(a^{-1}g) = (a * \overline{f_1})(g) = \overline{f_1}(g) = \overline{\phi}(a^{-1}gx).$$

Since g is arbitrary and G acts on X transitively we get that for every $y \in X$, $(a * \overline{\phi})(y) = \overline{\phi}(y)$. Since ϕ is limit aperiodic we have that $a \in Fix(X)$. But:

$$Fix(X) = \bigcap_{s \in S} Stab_G(s)$$

Thus, $a \in Fix(X) \subseteq Stab_G(x) = G_x$ contradiction.

$\underline{\text{Case 2)}}$ Suppose that $a \in G_x$. Let $\{h_k\}$ be a sequence of elements of G such that h_k^{-1} belongs to the coset $a_k G_x$. Thus $\delta_k = h_k a_k$ belongs to G_x. Taking a subsequence we may assume that there are the limits

$$\overline{\psi} = \lim_k \delta_k * \psi \qquad \text{and} \qquad \overline{f} = \lim_k h_k * f.$$

Notice that $\overline{\psi} \in \overline{Orb_{G_x}(\psi)}$. Let $h \in G_x$. Then there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have

$$\overline{\psi}(h) = (\delta_k * \psi)(h)$$

Also there exists a $k_1 \in \mathbb{N}$ such that for all $k \geq k_1$ we get:

$$\overline{f}|_{G_x}(h) = (h_k * f)(h)$$

Then for all $k \geq \max\{k_0, k_1\}$ we have: $\overline{f}|_{G_x}(h) = f(h_k^{-1}h)$. Hence,

$$(\overline{f}|_{G_x})_2(h) = \psi(a_k^{-1} h_k^{-1} h) = \psi((h_k a_k)^{-1} h) = ((h_k a_k) * \psi)(h) = (\delta_k * \psi)(h) = \overline{\psi}(h).$$

Notice now that:

$$a * \overline{\psi} = a * \overline{f}|_{G_x} = \overline{f}|_{G_x} = \overline{\psi}.$$

Thus $\overline{\psi}$ is periodic. Contradiction since ψ is limit aperiodic. This concludes the proof. □

The following is obvious.

Lemma 3.5. *If X is a G-space and H is any group acting on X such that the action of H factors through the action of G. Then if X is a limit aperiodic G-space then it is also a limit aperiodic H-space.*

Corollary 3.6. *If G and H are limit aperiodic groups and*

$$1 \to G \xrightarrow{\tau} E \xrightarrow{\pi} H \to 1$$

is a short exact sequence then E is also limit aperiodic.

Proof. Obviously E acts transitively on $E/G = H$ with left multiplication. By Lemma 3.5 H is a limit aperiodic E-space. Note that $Stab_E(e_H) = G$ is limit aperiodic. If we apply Theorem 3.4 we get the corollary. □

Obviously we obtain the following

Corollary 3.7. *If G is limit aperiodic and H is limit aperiodic then $G \times H$ is limit aperiodic.*

Corollary 3.8. *If H is a limit aperiodic group and $\theta : H \to H$ is a group automorphism then the HNN extension $\star_\theta H$ is limit aperiodic.*

Proof. We know that if $H =< S|T >$ where S is a set of generators and T is a set of relations then $G = \star_\theta H$ admits the following presentation

$$\star_\theta H =< S, t|T \cup \{t^{-1}xt = \theta(x), x \in S\} > .$$

Note that G acts transitively on the set G/H of all left cosets of H. Note that $G/H = \{t^i H \mid i \in \mathbb{Z}\} \cong \mathbb{Z}$. Thus, G acts on $\mathbb{Z}$ by translations with $Stab_G(\{H\}) = H$ [2]. We color the $\mathbb{Z}$ with a Morse-Thue sequence as in [1], i.e. $\phi : X \to \{0, 1\}$ with $\phi(t^i H) = m(|i|)$ where $m : \mathbb{N} \to \{0, 1\}$ is the Morse-Thue sequence [3],[4]. As it shown in [1] this map is limit aperiodic with respect. By Lemma 3.5 it is a limit aperiodic G-set. Theorem 3.4 completes the proof. □

In order to prove the fact that the free product of limit aperiodic groups is limit aperiodic we need the following notions. Let A and B be two groups. We will construct a set X such that the free product $G = A \star B$ acts on X freely and transitively and X is a limit aperiodic G-space. Let T_0 be the Bass-Serre tree associated with $A \star B$. We recall that the vertices of T_0 are left cosets $G/A \cup G/B$ and the vertices of the type xA, xB and only them form an edge $[xA, xB]$ in T_0. Thus the edges of T_0 are in the bijection with G. Note that G acts on T_0 by left multiplication. Let T be the barycentric subdivision of T_0 and let X be the set of the barycenters (of edges). We will identify the tree T with the set of its vertices. Then the group G acts by isometries on T yielding three orbits on the vertices $X = Orb_G(e)$, G/A and G/B. We regard T as a rooted tree with the root e. Let $\|x\| = d_T(x, e)$ denote the distance to the root.

Lemma 3.9. *Let A,B be limit aperiodic groups and let $G = A \star B$, X, T defined as above. Then T is a limit aperiodic G-set.*

Proof. Let $\pi : G \to A$ with $\pi(w) = \pi(a_1b_1a_2b_2...a_nb_n) = a_1a_2...a_n$ and $\theta : G \to B$ with $\theta(w) = \theta(a_1b_1a_2b_2...a_nb_n) = b_1b_2...b_n$. Clearly both π and θ are group homomorphisms. Let also $f_A : A \to F_A$ be the limit aperiodic map for the group A and $f_B : B \to F_B$ be the limit aperiodic map for group B. Also let $\nu : Z \to \{0, 1, 2\}$ be the variation of the Morse-Thue sequence which has no words WW (see for example [1]). Also fix e to be the vertex representing the identity element in T. Then consider a coloring of T as follows:

$$f : T \to \{0, 1, 2\} \times \{0, 1, 2\} \times (F_A \bigcup \{\alpha\}) \times (F_B \bigcup \{\beta\}) = F$$

where $f := (f_0, f_1, f_2, f_3)$ with: $f_0(x) = \nu(\|x\|)$, $f_1(x) = \|x\| mod 3$, $f_2(x) = f_A(\pi(x))$ if $x \in X$ and $f_2(x) = \alpha$ if $x \in T - X$.

Finally let $f_3(x) = f_B(\theta(x))$ if $x \in X$ and $f_3(x) = \beta$ if $x \in T - X$. The group G acts on the space of colorings F^T as follows:

$$\begin{aligned}(g * f)(x) &= ((g * f_0)(x), (g * f_1)(x), (\pi(g) * f_2)(x), (\theta(g) * f_3)(x) \\ &= (f_0(g^{-1}x), f_1(g^{-1}x), f_2(\pi^{-1}(g)x), f_3(\theta^{-1}(g)x)).\end{aligned}$$

Suppose that f is not a limit aperiodic map. Then there exists a coloring

$$\psi = (\psi_0, \psi_1, \psi_2, \psi_3)$$

such that $\psi \in \overline{Orb_G(f)}$ and ψ has a period $b \in G \setminus Fix(T)$. Let $\psi = \lim g_k * f$. Then $\psi_A = \lim \pi(g_k) * f_A$ has period $\pi(b)$. Indeed, for every $x \in A \subset G \cong X$ for large enough k,

$$\begin{aligned}(\pi(g_k) * f_A)(x) = f_A(\pi(g_k^{-1}x)) = f_2(g_k^{-1}x) = (g_k * f_2)(x) = (g_k * f_2)(bx) \\ = f_2(g_k^{-1}bx) = f_A(\pi(g_k^{-1}bx)) = (\pi(g_k) * f_A)(\pi(b)x).\end{aligned}$$

Similarly $\psi_B = \lim \theta(g_k) * f_B$ has a period $\theta(b)$. Thus, $\pi(b) = e_A$ and $\theta(b) = e_B$.

Notice that $(\psi_0, \psi_1) \in \overline{Orb_G(f_0, f_1)}$. Denote $\xi = (\psi_0, \psi_1)$ and $\phi = (f_0, f_1)$. Then ξ is a coloring of a simplicial tree (T) on which G acts by isometries. Moreover $\xi \in \overline{Orb_G(\phi)}$. From proposition 4, page 318 in [1] we have that $b * \xi \neq \xi$ for all $b \in G$ with unbounded orbit $\{b^k x_0 | k \in \mathbb{Z}\}$. This clearly implies that $b * \psi \neq \psi$ for every $b \in G$ with unbounded orbit. On the other hand ψ has period b and thus we have $\{b^k x_0 | k \in \mathbb{N}\}$ is bounded. This implies that b fixes a point in T. Call that point x_1. Since the action of G on X is free, $x_1 \notin X$. Thus, $x_1 \in G/A$ or $x_1 \in G/B$. Assume the later, $x_1 = wB$ for some $winG$. Since b fixes wB, $b = wb'w^{-1}$ for some $b' \in B \setminus \{e_B\}$. Then $\theta(b) = \theta(w)b'\theta(w)^{-1} \neq e_B$. Contradiction. □

Lemma 3.10. *Let $G = A \star B$, T, X, f, F as above. Then X is a limit aperiodic G-set.*

Proof. Note that in the rooted tree T every vertex $x \neq e$ has a unique predecessor denoted $pred(x)$. We define $f' : X \to F \times F$ as $f'(x) = (f|_X, \hat{f})$ where $\hat{f}(x) = f(pred(x))$. We show that f' is limit aperiodic.

Suppose that f' is not limit aperiodic. Then there exists a sequence $\{g_k\} \in G$ s.t.

$$\psi' = \lim_k g_k * f'$$

and $b \in Fix_G(X)$ with $b * \psi' = \psi'$. We may assume that there is the limit $\psi = \lim g_k * f$. In view of Lemma 3.9 it suffices to show that ψ is b-periodic.

It is b-periodic on X, so it suffice to check that it is b-periodic on $T \setminus X$. Let $z \in T \setminus X$. We check that $\psi(bz) = \psi(z)$. Since the root e lies in X, we may assume that $z \neq e$. Let $x_0 = pred(z)$ and let x_1 be such that $z = pred(x_1)$. We note that x_1 is not unique. So we fix one. Note that $x_0, x_1 \in X$. There is k_0 such that for $k \geq k_0$

$$\psi'(x_i) = (g_k * f')(x_i), \quad \psi'(bx_i) = (g_k * f')(bx_i), \quad i = 0, 1$$

and

$$\psi(z) = (g_k * f)(z), \quad \psi(bz) = (g_k * f)(bz).$$

Fix $k \geq k_0$. Since G acts on T by isometries the distance from $g_k^{-1}z$ to $g_k^{-1}x_i$, $i = 0, 1$ equals 1. There are three possibilities:

$$g_k^{-1}x_0 < g_k^{-1}z < g_k^{-1}x_1, \tag{1}$$

$$(2) \qquad g_k^{-1}x_1 < g_k^{-1}z < g_k^{-1}x_0,$$

and

$$(3) \qquad g_k^{-1}z < g_k^{-1}x_i,\ i = 0, 1.$$

We apply $g_k^{-1}bg_k$. In view of the fact that $f_1(g_k^{-1}x_i) = f_1(g_k^{-1}bx_i),\ i = 0, 1$ we obtain

$$g_k^{-1}bx_0 < g_k^{-1}bz < g_k^{-1}bx_1$$

in the case (1) and

$$g_k^{-1}bx_1 < g_k^{-1}bz < g_k^{-1}bx_0,$$

in the case (2). Then in the case (1)

$$f'(g_k^{-1}bx_1) = (g_k * f')(bx_1) = \psi'(bx_1) = \psi'(x_1) = (g_k * f')(x_1) = f'(g_k^{-1}x_1).$$

Hence $f_0(g_k^{-1}bx_1) = f_0(g_k^{-1}x_1)$. Therefore

$$f(g_k^{-1}bz) = f(g_k^{-1}z).$$

Thus,

$$\psi(bz) = (g_k * f)(bz) = f(g_k^{-1}bz) = f(g_k^{-1}z) = (g_k * f)(z) = \psi(z).$$

In the case (2) we consider x_0 instead of x_1.

In the case (3) $g_k^{-1}bz$ is the predecessor of either $g_k^{-1}bx_0$ or $g_k^{-1}bx_1$ (or both). Assume the first. Then from the b-periodicity of ψ' it follows that $f_0(g_k^{-1}bx_0) = f_0(g_k^{-1}x_0)$. Since $f_0(g_k^{-1}bx_0) = f(pred(g_k^{-1}bx_0))$ and $f_0(g_k^{-1}x_0) = f(pred(g_k^{-1}x_0))$, we obtain $\psi(bz) = f(g_k^{-1}bz) = f(g_k^{-1}z) = \psi(z)$. □

Theorem 3.11. *Let A,B be limit aperiodic groups. Then $G = A \star B$ is a limit aperiodic group.*

Proof. By Lemma 3.10 X is a limit aperiodic G-set. Note that G acts on X as above transitively, and $Stab_G(x_0) = \{e\}$ is a limit aperiodic group. By Theorem 3.4 we have that G is a limit aperiodic group. □

We continue with an example of a G-set which is not limit aperiodic.

Proposition 3.12. *Consider the group of isometries of the integers $Iso(\mathbb{Z})$. Let $s : \mathbb{Z} \to \mathbb{Z}$ with $s(n) = n + 1$ and $t : \mathbb{Z} \to \mathbb{Z}$ with $t(0) = 1$, $t(1) = 0$ and $t(n) = n$ if $n \neq 1$ and $n \neq 0$. Let $S =< s, t >$ then $\mathbb{N}$ is not a limit aperiodic S-set.*

Proof. Suppose that $\mathbb{N}$ was a limit aperiodic S-set. Then let F be a set of colors with $|F| < \infty$ and a map $f : \mathbb{N} \to F$ such that f is a limit aperiodic map under the action of S. Since $|F| < \infty$ and $|\mathbb{N}| = \infty$ there exists at least one $a \in F$ such that infinitely many $a_n \in \mathbb{N}$ have $f(a_n) = a$. Choose a strictly increasing sequence in $\mathbb{N}$ such that $f(a_n) = a$ for all $n \in \mathbb{N}$. Consider the following elements in S:

$$\begin{aligned} h_1 &= (1, a_1) \\ h_2 &= (1, a_1)(2, a_2) \\ &\dots \\ h_n &= (1, a_1)(2, a_2)\dots(n, a_n) \\ &\dots \end{aligned}$$

where (i, a_i) is the transposition that takes i to a_i. With s and t we can construct all the transpositions. Thus all a_i belong to S.

Clearly if $n \geq k$, $n, k \in \mathbb{N}$ we have that $h_n(k) = a_k$. Consider now the sequence $\{h_n^{-1} * f\}$ and take a converging subsequence. For convenience in notation let us keep the same indices for the subsequence. Thus if:

$$\psi = \lim_{n\to\infty} h_n^{-1} * f$$

we have that $\psi \in \overline{Orb_S(f)}$. Thus ψ has to be aperiodic. The claim is that $\psi(k) = a$ for all $k \in \mathbb{N}$ and thus ψ is clearly periodic which will lead to a contradiction. This is easy to see since let $k \in \mathbb{N}$. Then for that k there exists an n_1 such that for all $n \geq n_1$ we have that $\psi(k) = (h_n^{-1} * f)(k)$. Thus for $n = \max\{k, n_1\}$ we have that:

$$\psi(k) = (h_n^{-1} * f)(k) = f(h_n k)) = f(a_k) = a.$$

This concludes the proof. □

4. Limit Aperiodic Subgroups

The fact that there exist limit aperiodic G-spaces and others that are not limit aperiodic, when G acts transitively on them, turns out to be a question on the subgroups of G. We introduce the definition of a limit aperiodic subgroup to investigate the limit aperiodic G-spaces.

Definition 4.1. Fix a group G. Let $H < G$ a subgroup of G. We will call H *limit aperiodic subgroup in* G if the space of quotients $\frac{G}{H}$ is a limit aperiodic G-set.

Remark 4.2. *Clearly if* G *acts transitively on a space* X *then* X *is a limit aperiodic* G*-space if and only if* $Stab_G\{x_0\}$ *is a limit aperiodic subgroup of* G *where* $X = Orb_G(x_0)$.

Proposition 4.3. *Suppose* N *is a normal subgroup of* G. *If* $H \simeq \frac{G}{N}$ *is a limit aperiodic group, then* N *is a limit aperiodic subgroup of* G.

Proof. Notice that

$$1 \to N \to G \to H \to 1$$

is a short exact sequence. Since H is a limit aperiodic group, it is a limit aperiodic H-space. Since the action of G on H factors through H, we get that H is a limit aperiodic G-space. Thus $\frac{G}{N}$ is a limit aperiodic G-space. □

Proposition 4.4. *All finite spaces are limit aperiodic* G*-sets.*

Proof. This is true since let $X = \{x_1, x_2, \ldots, x_n\}$. Consider the coloring $f : X \to \{1, 2, ...|X|\}$ were $f(x_i) = x_i$. This map is clearly a limit aperiodic map. □

Corollary 4.5. *All finite groups are limit aperiodic.*

Proof. Since G is finite it is a finite space. So from the previous remark it is a limit aperiodic G-set and thus a limit aperiodic group. □

Corollary 4.6. *Every subgroup of finite index is a limit aperiodic subgroup.*

Proof. Since H has finite index, $X = \frac{G}{H}$ is a finite space. By the proposition above $\frac{G}{H}$ is a limit aperiodic G-set. □

Proposition 4.7. *If* N *is a limit aperiodic subgroup of* G *and* $N < H$ *where* $|\frac{H}{N}| = n < \infty$ *then* H *is a limit aperiodic subgroup of* G.

Proof. Let $f : \frac{G}{N} \to F$ be the limit aperiodic map for $\frac{G}{N}$ as a G-space. Let $a_{1},_{2}, ...a_n$ be a set of representatives for $\frac{H}{N}$. Define a map $F : \frac{G}{H} \to F^n$ with:

$$F(bH) = (f(ba_1N), f(ba_2N), ..., f(ba_nN))$$

Suppose that F was not limit aperiodic. Then there exists a sequence $\{g_k\} \in G$ and a $c \in G$, $c \notin Fix_G(G/H)$ such that if $\Psi = \lim g_k * F$ then $\Psi = c * \Psi$. Consider $\psi = \lim g_k * f$(choose a converging subsequence so that the limit exists and rename the indices). From the definition of Ψ and the fact that $\Psi = c * \Psi$ we get that $\psi = c * \psi$. But $\psi \in \overline{Orb_G(f)}$. Since f is limit aperiodic we have that $c \in Fix_G(G/N)$. Clearly $Fix_G(G/N) \subseteq Fix_G(G/H)$. Thus $c \in Fix_G(G/H)$ which is a contradiction. Thus F is limit aperiodic and $\frac{G}{H}$ is a limit aperiodic G-set which forced H to be a limit aperiodic subgroup □

Corollary 4.8. *If G is a limit aperiodic group and H is a finite subgroup of G then H is a limit aperiodic subgroup of G.*

Proof. Since G is a limit aperiodic group it is also a limit aperiodic G-space. Consider $N = \{e\}$. Clearly $\frac{G}{N} \simeq G$. Thus $\frac{G}{N}$ is a limit aperiodic G-set. So $N = \{e\}$ is a limit aperiodic subgroup of G. Notice that $N < H$ and $|\frac{H}{N}| = n = |H| < \infty$. Thus by the previous corollary H is a limit aperiodic subgroup of G. □

Assertion 4.9. In the previous proof we asserted the following equivalence. A group G is limit aperiodic if and only if $\{e\}$ is a limit aperiodic subgroup of G.

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